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Pearson Global Indicator and the Quantum Mechanics of *p*-Dimensional Systems

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We extend the results of Guiasu (1992a) to p-dimensional systems. We use quantum mechanics in order to extend the basic mathematical model from Guiasu (1992a) to systems with more dimensions. Two multidimensional quantum systems are presented as applications of the mathematical results (the p-dimensional isotropic harmonic oscillator and the free particle in a p-dimensional box).

INTRODUCTION

Guiasu (1992a) derives Schrödinger's equation considering the optimum Pearson function to be the wave function of the system. The optimum Pearson function χ^* minimizes the Pearson global indicator subject to the constraints represented by some given mean fluctuations.

The steady-state condition of a quantum system is characterized by the probability distribution u that maximizes Shannon's entropy and complies with the restrictions represented by some mean values known from macroscopic measurements. If random fluctuations alter the steady-state condition, then the most unbiased probability distribution u no longer correctly describes the quantum system. Such a change can be detected by assigning a sequence of orthonormal functions with the weight u. As long as the system is in the steady-state condition, the mean value of each orthonormal function is equal to zero. If at least one of these mean values is not zero, then the steady state of the system has been altered.

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Further we estimate the probability distribution of the fluctuations subject to the mean values detected and which minimizes Pearson global indicator $\langle\chi^2\rangle$ (Pearson mean deviation). We estimate therefore the probability distribution of the fluctuations which is the closest to the steady-state probability distribution and subject to some given mean values of the fluctuations which are obtained by calculating the mean values of the orthonormal functions with the weight u assigned to the quantum system.

The optimum Pearson function χ^* that minimizes the Pearson indicator $\langle\chi^2\rangle$ satisfies Schrödinger's equation if we consider classical quantization rules.

This probabilistic model presented in Guiasu (1992a) is applied to the one-dimensional harmonic oscillator, the free particle in the box $[0, a]$, and the hydrogen atom. The extension of the results from Guiasu (1992a) to p-dimensional systems is based on some important results of quantum mechanics.

1. OPTIMUM PEARSON FUNCTION χ^*

If f and g are two square-integrable functions on $D \subset R\,[f, g \in L^2(D)]$, we define the scalar product between the two functions with the weight u as follows:

$$\langle f|g\rangle_u = \int_D f(x)g(x)u(x)\,dx$$

In the above equality, the integral is considered with respect to the Lebesgue measure on the real axis. If $u(x) = 1$, $\forall\, x \in D$, then we write $\langle f|g\rangle_1 = \langle f|g\rangle$.

We consider a p-dimensional quantum system and let $X_1, X_2, \ldots, X_p$ be the random variables describing the p characteristics of the quantum system, $D_1, D_2, \ldots, D_p$ be the ranges of the above-mentioned random variables, and $u_1, u_2, \ldots, u_p$ be the probability distributions that describe the random variables in the steady–state.

We emphasize that $X_1, X_2, \ldots, X_p$ are independent random variables and the probability distribution $u = u_1u_2 \ldots u_p$ describes the p-dimensional system in the steady–state.

Let us also consider the following p sequences of orthonormal functions:

$\Omega^{(1)} = \{U^{(1)}_{n_1}, n_1 = 0, 1, 2, \ldots\}$ is a sequence of orthonormal functions in $L^2(D_1)$ with the weight u_1 and $U^{(1)}_0 = 1$.

$\Omega^{(2)} = \{U^{(2)}_{n_2}, n_2 = 0, 1, 2, \ldots\}$ is a sequence of orthonormal functions in $L^2(D_2)$ with the weight u_2 and $U^{(2)}_0 = 1$.

$\Omega^{(p)} = \{U_{n_p}^{(p)}, n_p = 0, 1, 2, \ldots\}$ is a sequence of orthonormal functions in $L^2(D_p)$ with the weight u_p and $U_0^{(p)} = 1$.

We have

$$\langle U_{n_j}^{(j)} \mid U_{n_j'}^{(j)} \rangle_{u_j} = \delta_{n_j}^{n_j'}, \quad j = \overline{1, p}, \qquad \text{where} \quad \delta_{n_j}^{n_j'} = \begin{cases} 1 & \text{for } n_j = n_j' \\ 0 & \text{for } n_j \neq n_j' \end{cases}$$

We also consider the system of functions

$$\Omega = \{U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)}, n_1 = 0, 1, 2, \ldots, n_2 = 0, 1, 2, \ldots, \ldots, n_p = 0, 1, 2, \ldots\}$$

This system of functions is orthonormal in $L^2(D_1 \times D_2 \times \ldots \times D_p)$ with the weight $u = u_1 u_2 \ldots u_p$. Indeed

$$\begin{aligned} &\langle U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)} \big| U_{n_1'}^{(1)} U_{n_2'}^{(2)} \ldots U_{n_p'}^{(p)} \rangle_{u_1 u_2 \ldots u_p} \\ &= \langle U_{n_1}^{(1)} \big| U_{n_1'}^{(1)} \rangle_{u_1} \langle U_{n_2}^{(2)} \big| U_{n_2'}^{(2)} \rangle_{u_2} \ldots \langle U_{n_p}^{(p)} \big| U_{n_p'}^{(p)} \rangle_{u_p} = \delta_{n_1}^{n_1'} \delta_{n_2}^{n_2'} \ldots \delta_{n_p}^{n_p'} \end{aligned} \quad (1)$$

The above relation is a consequence of Fubini's theorem in the case of the integrate equal to a product of functions depending each on only one variable.

If random fluctuations alter the steady-state condition of the system, then the random variables $X_1, X_2, \ldots, X_p$ become dependent and the probability distribution $u = u_1 u_2 \ldots u_p$ no longer accurately describes the behavior of the quantum system. Let f be the new common probability distribution that describes the quantum system. Let us consider fluctuations of type $U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)}$ with $n_1 = \overline{1, N_1}$, $n_2 = \overline{1, N_2}$, $\ldots$, $n_p = \overline{1, N_p}$, where N_j, $j = \overline{1, p}$, are fixed natural numbers.

Further let us assume that the induced mean fluctuations are given, i.e.,

$$\langle U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)} \big| f \rangle = C_{n_1 n_2 \ldots n_p} \qquad \text{with} \quad n_j = \overline{1, N_j}, \quad j = \overline{1, p} \quad (2)$$

At this point we determine the probability distribution f^* which is the closest to $u = u_1 u_2 \ldots u_p$ subject to the restrictions from (2). According to Guiasu (1992a), the closeness is measured by Pearson's global indicator:

$$\langle \chi^2(f{:}u) \rangle = \langle \chi^2(f{:}u) \big| 1 \rangle = \left\langle \frac{(f-u)^2}{u} \middle| 1 \right\rangle = \left\langle \left(\frac{f}{u} - 1 \right)^2 \middle| u \right\rangle \quad (3)$$

Therefore f^* is the solution of the variational problem

$$\begin{cases} \min_f \langle \chi^2(f:u) \rangle \\ \langle U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)} \big| f \rangle = C_{n_1 n_2 \ldots n_p}, \qquad n_j = \overline{1, N_j}, j = \overline{1, p} \end{cases} \tag{4}$$

Extending the results from Guiasu (1992b), Guiasu *et al.*, (1982), and Kullbach and Leibler (1951) to the p-dimensional case, we have

$$f^* = u \cdot \left[1 + \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \right] \tag{5}$$

Since $1 \in \Omega^{(j)}$, $j = \overline{1, p}$, it can be easily deduced that f^* is a probability distribution.

From (3) and (5) we get the optimum Pearson function χ^*:

$$\begin{aligned} \chi^* &= \chi(f^*:u) \\ &= \left(\frac{f^*}{u} - 1 \right) \cdot \sqrt{u} \\ &= \sqrt{u} \cdot \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \end{aligned} \tag{6}$$

We shall prove further that the probability distribution f^* from (5) minimizes the Pearson global indicator from (3). Indeed, if the orthonormal sequence Ω is complete in $L^2(D_1 \times D_2 \times \ldots \times D_p)$, we can write the following Fourier series (Precupanu, 1976):

$$\frac{f}{u} = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_p=1}^{\infty} \left\langle \frac{f}{u} \middle| U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)} \right\rangle_u \cdot U_{n_1}^{(1)} U_{n_2}^{(2)} \ldots U_{n_p}^{(p)}$$

Hence

$$\begin{aligned} f &= u \cdot \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_p=1}^{\infty} \langle f \big| U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \rangle \cdot U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \\ &= u \cdot \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} \langle f \big| U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \rangle \cdot U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} + u \cdot W \\ &= u \cdot \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} \cdot U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} + u \cdot W \\ &= f^* + u \cdot (W - 1) \end{aligned}$$

where W is a finite sum of p-uple series in which at least one summation index n_j is greater than N_j.

We have

$$\langle f^* - u | W \rangle = 0 \tag{7}$$

because at least one orthonormal function $U_{n_j}^{(j)}$ from the expression of $f^* - u$ ($n_j \leq N_j$) is different from any orthonormal function $U_{n_j}^{(j)}$ from the expression of W ($n_j > N_j$). We shall calculate the Pearson global indicator taking into consideration (7) and the relation$\langle f^* - u | 1 \rangle = \langle f^* | 1 \rangle - \langle u | 1 \rangle = 1 - 1 = 0$, which is true since f^* and u are two probability distributions:

$$\begin{aligned}
\langle \chi^2(f{:}u) \rangle &= \left\langle \left(\frac{f}{u} - 1 \right)^2 \middle| u \right\rangle = \left\langle \left(\frac{f^* + u(W-1)}{u} - 1 \right)^2 \middle| u \right\rangle \\
&= \left\langle \left(\frac{f^*}{u} - 1 \right)^2 \middle| u \right\rangle + \langle (W-1)^2 | u \rangle \geq \left\langle \left(\frac{f^*}{u} - 1 \right)^2 \middle| u \right\rangle \\
&= \langle \chi^2(f^*{:}u) \rangle \quad \text{QED}
\end{aligned}$$

Considering (6) and (1), the normed optimum Pearson function χ^* becomes

$$\Psi^* = \left(\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p}^2 \right)^{-1/2} \sqrt{u_1 u_2 \ldots u_p}$$
$$\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \tag{8}$$

Thus $(\Psi^*)^2$ is a probability distribution induced by the minimization of the Pearson global indicator and which expresses the deviation of the quantum system from the steady–state due to the fluctuations $U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)}$ having the mean values $C_{n_1 n_2 \ldots n_p}$. Since Ψ^* is a normed square-integrable function, it satisfies the necessary conditions to be a wave function (Messiah, 1973).

2. HEISENBERG'S UNCERTAINTY RELATIONS

For the function Ψ^* determined in (8) we shall prove Heisenberg's uncertainty relations in the manner done in Messiah (1973). In order to do that we will introduce the momentum operator.

In quantum mechanics to any dynamic variable a (position coordinate, momentum, energy, etc.) there is assigned a linear operator A, and the mean value taken by the dynamic variable a in the state defined by the function Ψ^* is given by the relation

$$\bar{a} = \langle \Psi^* | A\Psi^* \rangle \tag{9}$$

So, according to the principle of correspondence (Messiah, 1973), to the momentum p_j there is assigned the linear operator $P_j = -i\hbar\, \partial/\partial x_j$ (the momentum operator), where x_j is a position coordinate, $i^2 = -1$, and $\hbar = h/2\pi$ with h Planck's constant, and $j = \overline{1, p}$. We have the following two formulas:

$$\overline{p_j} = \langle \Psi^* | P_j\Psi^* \rangle = \left\langle \Psi^* \middle| -i\hbar \frac{\partial}{\partial x_j} \Psi^* \right\rangle, \qquad j = \overline{1, p} \tag{10}$$

$$\overline{p_j^2} = \langle \Psi^* | P_j^2\Psi^* \rangle = \left\langle \Psi^* \middle| -\hbar^2 \frac{\partial^2}{\partial x_j^2} \Psi^* \right\rangle, \qquad j = \overline{1, p} \tag{11}$$

We build the operator

$$F_j = \Delta x_j + \lambda i\, \Delta p_j \tag{12}$$

where $\Delta x_j = x_j - \overline{x_j}$, $\Delta p_j = p_j - \overline{p_j} = -i\hbar\, \partial/\partial x_j - \overline{p_j}$, $\lambda \in R$.

We use the following notations for the variances of the distributions of the coordinate x_j and momentum p_j:

$$\sigma x_j = \sqrt{\overline{(\Delta x_j)^2}} = \sqrt{\overline{x_j^2} - \overline{x_j}^2}, \qquad \sigma p_j = \sqrt{\overline{(\Delta p_j)^2}} = \sqrt{\overline{p_j^2} - \overline{p_j}^2}, \qquad j = \overline{1, p}$$

In order to deduce Heisenberg's uncertainty relations for the function Ψ^*, we will start from the following positively defined expression:

$$I_j(\lambda) = \int_{D_1}\int_{D_2} \cdots \int_{D_p} \left| F_j\Psi^* \right|^2 dx_1\, dx_2 \ldots dx_p \geq 0 \qquad \forall\lambda \in R, \quad j = \overline{1, p} \tag{13}$$

Integrating by parts in the relation (13) and taking into account (10), (11), and the fact that the function Ψ^* is square-integrable, we obtain

$$I_j(\lambda) = (\sigma x_j)^2 - \lambda\hbar + \lambda^2(\sigma p_j)^2, \qquad j = \overline{1, p} \tag{14}$$

Therefore

$$(\sigma x_j)^2 - \lambda\hbar + \lambda^2(\sigma p_j)^2 \geq 0 \qquad \forall\lambda \in R, \quad j = \overline{1, p}$$
$$\Leftrightarrow \hbar^2 - 4(\sigma x_j)^2(\sigma p_j)^2 \leq 0, \quad j = \overline{1, p}$$
$$\Rightarrow \sigma x_j \cdot \sigma p_j \geq \frac{\hbar}{2}, \qquad j = \overline{1, p} \tag{15}$$

Therefore, Heisenberg's uncertainty relations, $\sigma x_j \cdot \sigma p_j \geq \hbar/2$, $j = \overline{1, p}$, are a consequence of the properties of the function Ψ^* (square-integrable) and the manner in which the momentum operator is defined. We underline the fact that up to this moment no reference has been made to Schrödinger's equation; thus Heisenberg's relations are valid for the function Ψ^* found in Section 1 and for which we have not shown yet that it is a wave function: Ψ^* does satisfy the Schrödinger equation only if the classical quantization rules hold. This fact will be emphasized in Sections 3 and 4.

3. SCHRÖDINGER'S EQUATION

In this section we will show that the function Ψ^* from (8) behaves like a wave function, i.e., it satisfies Schrödinger's equation, if we use classical quantization rules. This result is demonstrated in Guiasu (1992a) for the one-dimensional case and we will extend it to the p-dimensional case in this paper.

The Schrödinger equation for a p-dimensional system is

$$\Delta\Psi - \frac{1}{\hbar^2} k\Psi = 0 \tag{16}$$

where Ψ is the wave function that describes the behavior of the quantum system [any macroscopic measure can be calculated as a mean value according to (9)], $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_p^2$, and $k = 2m(V - E)$ with m the mass of the quantum system, V the potential energy, and E the total energy.

By placing Ψ^* from (8) into the Schrödinger equation (16), we get

$$\begin{aligned} &\sum_{j=1}^{p} \left\{ \sqrt{u} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} \right. \\ &\quad \times \left. \left[-\frac{(u_j')^2}{4u_j^2} + \frac{u_j''}{2u_j} + \frac{u_j'}{u_j} \frac{(U_{n_j}^{(j)})'}{U_{n_j}^{(j)}} + \frac{(U_{n_j}^{(j)})''}{U_{n_j}^{(j)}} \right] \right\} \\ &\quad - \frac{1}{\hbar^2} k \sqrt{u} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)} = 0 \end{aligned} \tag{17}$$

In what follows we shall assume that the parameter k complies with the condition

$$k = \sum_{j=1}^{p} k_j \tag{18}$$

where k_j depends only on the variable x_j, i.e., the constant k can be written as a sum of components each of which depends on only one variable.

According to Messiah (1973), condition (18) is equivalent to

$$V = \sum_{j=1}^{p} V_j \qquad \text{where} \quad V_j = V_j(x_j) \tag{19}$$

[This condition is taken from quantum mechanics and it allows us to develop further the mathematical calculations. It applies to a rather limited number of quantum systems. However, these quantum systems are of essential importance from a theoretical point of view because condition (18) is necessary for finding exact solutions for Schrödinger's equation. Despite the restricted applicability, the mathematical model presented in this paper yields great relevance as will be seen from the examples presented in Section 4.]

Assuming that condition (18) is satisfied, relation (17) becomes

$$\sqrt{u} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \cdots \sum_{n_p=1}^{N_p} C_{n_1 n_2 \ldots n_p} U_{n_1}^{(1)} U_{n_2}^{(2)} \cdots U_{n_p}^{(p)}$$
$$\times \sum_{j=1}^{p} \left[-\frac{(u_j')^2}{4u_j^2} + \frac{u_j''}{2u_i} + \frac{u_j'}{u_i} \frac{(U_{nj}^{(j)})'}{U_{nj}^{(j)}} + \frac{(U_{nj}^{(j)})''}{U_{nj}^{(j)}} - \frac{1}{\hbar^2} k_j \right] = 0 \tag{20}$$

Hence

$$u_j^2 (U_{nj}^{(j)})'' + u_j u_j' (U_{nj}^{(j)})' + \left[\frac{1}{2} u_j u_j'' - \frac{1}{4} (u_j')^2 - \frac{1}{\hbar^2} k_j u_j^2 \right] \cdot U_{nj}^{(j)} = 0,$$
$$j = \overline{1, p} \tag{21}$$

For any orthonormal set $\Omega^{(j)}$ of polynomials with the weight u_j the following second-order differential equation is satisfied (Guiasu, 1992a):

$$g_2^{(j)}(x_j)[U_{nj}^{(j)}(x_j)]'' + g_1^{(j)}(x_j)[U_{nj}^{(j)}(x_i)]' + g_0^{(j)}(x_j) U_{nj}^{(j)}(x_j) = 0$$
$$n_j = \overline{1, N_j}, \qquad j = \overline{1, p} \tag{22}$$

Thus Ψ^* satisfies the Schrödinger equation if

$$g_2^{(j)} = u_j^2, \qquad g_1^{(j)} = u_j u_j',$$
$$g_0^{(j)} = \frac{1}{2} u_j u_j'' - \frac{1}{4} (u_j')^2 - \frac{1}{\hbar^2} k_j u_j^2, \qquad j = \overline{1, p} \tag{23}$$

In Section 4 we shall see that, in the particular cases of two p-dimensional quantum systems, the relations (23) are satisfied if we take into account the classical quantization rules.

We point out that the results obtained above are analogous to those for the one-dimensional case presented in Guiasu (1992a).

4. EXAMPLES OF *p*-DIMENSIONAL QUANTUM SYSTEMS

We shall discuss two examples of p-dimensional quantum systems and we will estimate the wave function Ψ^* from (8). We will also show that Ψ^* satisfies Schrödinger's equation if we consider the classical quantization rules. For the two quantum systems presented we will check that the condition (18) is satisfied.

Let us consider X_j, $j = \overline{1, p}$, the position coordinates of a quantum particle of mass m, and D_j, $j = 1, p$, the ranges of the random variables X_j.

4.1. The *p*-Dimensional Isotropic Harmonic Oscillator

For the p-dimensional isotropic harmonic oscillator the elongation X_j ranges over $D_j = (-\infty, +\infty)$. Let μ_j be the mean value of X_j and let σ_j^2 be the variance of X_j. The steady-state condition that maximizes the entropy subject to the mean values μ_j and σ_j^2 is described by the normal distribution (Guiasu, 1977)

$$u_j(x_j) = \frac{1}{\sigma_j \sqrt{2\pi}} e^{-(x_j-\mu_j)^2/2\sigma_j^2}, \qquad -\infty < x_j < +\infty \tag{24}$$

The corresponding orthonormal set of polynomials with the weight u_j is (Guiasu, 1992a)

$$U_{n_j}^{(j)}(x_j) = \frac{1}{\sqrt{2^{n_j} n_j!}} H_{n_j}\left(\frac{x_j - \mu_j}{\sigma_j \sqrt{2}}\right), \qquad n_j = 0, 1, \ldots \tag{25}$$

where $H_n(x)$ is the Hermite polynomial of degree n (Teodorescu and Olariu, 1978).

If random fluctuations alter the steady–state, then the induced mean fluctuations are not equal to zero. Let us assume that only $Cn_1 n_2 \ldots n_p \neq 0$, the rest of the induced mean fluctuations being equal to zero. Then the normed optimum Pearson function from (8) becomes

$$\Psi^*_{n_1 n_2 \ldots n_p}(x_1, x_2, \ldots, x_p)$$
$$= \prod_{j=1}^{p} \left[\frac{1}{\sqrt{\sigma_j \sqrt{2\pi}}} e^{-(x_j-\mu_j)^2/4\sigma_j^2} \frac{1}{\sqrt{2^{n_j} n_j!}} H_{n_j}\left(\frac{x_j - \mu_j}{\sigma_j \sqrt{2}}\right) \right] \tag{26}$$

Considering $\mu_j = 0$, $\sigma_j^2 = \sigma^2$, $j = \overline{1, p}$, the function derived in (26) satisfies the partial differential equation (Guiasu, 1992a)

$$\Delta\Psi^*_{n_1 n_2 \ldots n_p} + \frac{1}{\sigma^2}\left[\left(n_1 + n_2 + \ldots + n_p + \frac{p}{2}\right) - \frac{x_1^2 + x_2^2 + \ldots + x_p^2}{4\sigma^2}\right]\Psi^*_{n_1 n_2 \ldots n_p} = 0 \tag{27}$$

For the p-dimensional isotropic harmonic oscillator (Messiah, 1973) the potential energy is given by

$$V = \frac{1}{2} m\omega^2 r^2 = \frac{1}{2} m\omega^2 (x_1^2 + x_2^2 + \cdots + x_p^2) = \sum_{j=1}^{p} \frac{1}{2} m\omega^2 x_j^2 = \sum_{j=1}^{p} V_j$$

$$V_j = \frac{1}{2} m\omega^2 x_j^2$$

where ω is the pulsation of the classical oscillator (constant). Therefore the condition (18) is satisfied. The Schrödinger equation (16) becomes

$$\Delta\Psi + \frac{2m}{\hbar^2}\left(E - \sum_{j=1}^{p} \frac{1}{2} m\omega^2 x_j^2\right)\Psi = 0 \tag{28}$$

Equation (27) becomes Schrödinger's equation (28) if we assume the quantization rules from quantum mechanics (Messiah, 1973):

$$\sigma^2 = \frac{\hbar}{2m\omega}, \qquad E = \hbar\omega\left(n_1 + n_2 + \cdots + n_p + \frac{p}{2}\right) \tag{29}$$

In this p-dimensional case, the wave function $\Psi(x_1, x_2, \ldots, x_p) = \Psi^*_{n_1 n_2 \ldots n_p}(x_1, x_2, \ldots, x_p)$ from (26) depends on p quantum numbers $n_1, n_2, \ldots, n_p$ that can take any integer value between 0 and $+\infty$, but the corresponding value of the energy $E = \hbar\omega(n_1 + n_2 + \cdots + n_p + p/2)$ depends only on the sum $n = n_1 + n_2 + \cdots + n_p$. For a given value of the sum n there exist $C_{n+p-1}^{p-1} = (n + p - 1)!/[n!\,(p - 1)!]$ different possible values for the sequence of integer numbers $n_1, n_2, \ldots, n_p$. Therefore the value of the energy $E_n = \hbar\omega(n + p/2)$ has an order of degeneracy equal to C_{n+p-1}^{p-1}.

4.2. The Free Particle in the p-Dimensional Box

In this case the random variable X_j ranges over $D_j = [0, a_j]$. Because the particle is supposed to be free, there are no other constraints imposed on the random variable X_j. The maximization of the entropy is satisfied by the uniform distribution (Guiasu, 1977)

$$u_j(x_j) = \frac{1}{a_j}, \qquad 0 \le x_j \le a_j \tag{30}$$

The corresponding orthonormal set of functions with the weight u_j is (Guiasu, 1992a)

$$U_0^{(j)}(x_j) = 1, \qquad U_{n_j}^{(j)}(x_j) = \sqrt{2} \sin \frac{n_j \pi x_j}{a_j}, \qquad n_j = 1, 2, \ldots \tag{31}$$

The steady–state described by $u = \prod_{j=1}^{p} u_j$ is changed by random fluctuations. Assuming further that only $C_{n_1 n_2 \ldots n_p} \neq 0$, all the rest of the induced fluctuations being equal to zero, the normed optimum Pearson function from (8) becomes

$$\Psi^*_{n_1 n_2 \ldots n_p}(x_1, x_2, \ldots, x_p) = \prod_{j=1}^{p} \sqrt{\frac{2}{a_j}} \sin \frac{n_j \pi x_j}{a_j} \tag{32}$$

The function from the previous relation satisfies the partial differential equation (Guiasu, 1992a)

$$\Delta \Psi^*_{n_1 n_2 \ldots n_p} + \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \ldots + \frac{n_p^2}{a_p^2} \right) \pi^2 \Psi^*_{n_1 n_2 \ldots n_p} = 0 \tag{33}$$

For the free particle in the *p*-dimensional box (Messiah, 1973) the potential energy is given by

$$V = 0, \quad 0 \le x_j \le a_j, \quad j = \overline{1, p}$$

or

$$V = \sum_{j=1}^{p} V_j, \quad V_j = 0, \quad 0 \le x_j \le a_j$$

Thus the condition (18) is satisfied. The Schrödinger equation (16) becomes

$$\Delta \Psi + \frac{2m}{\hbar^2} E \Psi = 0, \qquad 0 \le x_j \le a_j, \qquad j = \overline{1, p} \tag{34}$$

Equation (33) becomes Schrödinger's equation (34) if we assume the quantization rules from quantum mechanics (Messiah, 1973):

$$E = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \ldots + \frac{n_p^2}{a_p^2} \right) \tag{35}$$

The value of the energy E is not degenerate in general. If $a_j = a$, $j = \overline{1, p}$, i.e., the particle is in a *p*-dimensional cube, then the value of the energy $E = (\hbar^2\pi^2/2ma^2)(n_1^2 + n_2^2 + \ldots + n_p^2)$ has an order of degeneracy equal to $p!$.

5. CONCLUSIONS

In this paper, the wave function that describes a p-dimensional quantum system is deduced from a variational problem implying the minimization of the Pearson global indicator. The Schrödinger equation is obtained as a consequence. In this way, interpreting the square of the absolute value of the wave function as the probability distribution of the position coordinates of a quantum system yields a much larger significance in the context of information theory.

Heisenberg's uncertainty relations are satisfied by the optimum Person function before it satisfies Schrödinger's equation.

The normed optimum Pearson function behaves like a wave function, i.e., it satisfies the Schrödinger equation if we assume the quantization rules from quantum mechanics. As significantly expressed in Guiasu (1992a), the classical quantization rules seem to be "*the bridge between the unbiased probabilistic model built up and some physical characteristics of the quantum system involved.*"

The applicability of the theory is restricted to those quantum systems satisfying condition (18). The extension to quantum systems with no exact solution of the Schrödinger equation remains an open problem.

In the present paper, the applicability of the mathematical model is proven for two p-dimensional quantum systems of great interest: the p-dimensional isotropic harmonic oscillator and the free particle in a p-dimensional box.

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